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ON CERTAIN MATHEMATICAL QUESTIONS OF THE THEORY OF INCOMPRESSIBLE VISCOPLASTIC MEDIA

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There is given a proof of the equivalence of the differential and variational formulations of problems concerning the motion of a viscoplastic medium in the presence of domains of the rigid state of the medium and flow domains. The agreement between the upper bounds of the static limit load coefficients and the lower bound of the kinematic limit load coefficients results for a rigidly plastic body from the proof proposed.

Only the inequality between these bounds was known earlier in the general case. The equivalence of the corresponding formulations of the problems was known earlier in the case of nonlinearly viscous fluids possessing a dissipative potential (it follows directly from general theorems of the calculus of variations). The correctness of the formulations of the problems in differential form was studied in [1] by the method of variational inequalities for several particular cases of the motion of a viscoplastic medium with a dissipative Mises potential.

1. Functionals of integral type and their subdifferentials. Let $\Phi(e)$ be a convex, finite functional in the Banach space B, and let E be a closed linear set in B, i.e. $E = e_0 + H$ where e_0 is some element of E and H is a closed subspace in B. A linear, continuous functional $L(e)$ is called supporting to $\Phi(e)$ on E in e if

$$\Phi(e+h) - \Phi(e) \geqslant \langle L(e), h\rangle, \quad \forall h \in H, \quad L \in B^* \tag{1.1}$$

Here B^* is a space of linear continuous functionals on B, and $\langle L, g\rangle$ is the value of the functional L from B^* on g from B.

It is known (see [2]) that a convex, continuous functional has a support functional on B in any element e from B. The totality of all support functionals in e on E is called a subdifferential of the functional Φ in e on E and is denoted by $\partial\Phi(e)$ (see [3]). Therefore, a multivalued operator A is defined

$$A : e \in E \subset B \to \partial\Phi(e) \subset B^*$$

Lemma 1.1. The operator A is monotonic, i.e.,

$$\langle L(e_1) - L(e_2), e_1 - e_2\rangle \geqslant 0, \quad \forall L(e_i) \in \partial\Phi(e_i), \quad i = 1, 2 \tag{1.2}$$

P r o o f. Writing the inequality (1.1) for $e = e_1$, $h = e_2 - e_1$ and for $e = e_2$, $h = e_1 - e_2$ and then adding, we obtain (1.2).

Let u be an element from B and v_n, $n = 1,2, \ldots$ a compact set in B. Let V denote the following set of elements:

$$v_{n,k} = (v_n + (2^k - 1)\ u) \ /\ 2^k, \quad n = 1,\ 2, \ldots;\ k = 0,\ 1,\ 2, \ldots$$

L e m m a 1.2. For any v from V and some $L\ (v)$ from $\partial\Phi\ (v)$, let the following inequality be satisfied

$$\langle \chi - L\ (v),\ u - v \rangle \geqslant 0 \tag{1.3}$$

Then the functional χ enters into $\partial\Phi\ (u)$.

P r o o f. We find from the definition of $\partial\Phi$ and (1.3)

$$\langle \chi,\ v - u \rangle \leqslant \langle L\ (v),\ v - u \rangle \leqslant \Phi\ (2v - u) - \Phi\ (v) \tag{1.4}$$

Substituting the elements $v_{n,k}$, successively into (1.4) for $k = 0,\ 1,\ 2, \ldots$ and adding the inequalities obtained, we find

$$\langle \chi,\ 2\ (v_n - u) \rangle \leqslant \Phi\ (u + 2\ (v_n - u)) - \Phi\ (u) \tag{1.5}$$

The assertion of the lemma follows from (1.5) and the density of the elements v_n.

Let us note still another property of the subdifferential (the Morrow — Rockafellar theorem) which will be used below. Let $\Psi\ (e)$ have the form $\Psi\ (e) = \Phi_1\ (e) + \Phi_2\ (e)$, where $\Phi_i\ (e)$ are convex, continuous functionals in B. Then $\partial\Psi\ (e) = \partial\Phi_1\ (e) + \partial\Phi_2\ (e)$ (see [3]). In particular, if

$$\Psi\ (e) = \Phi\ (e) + \langle f,\ e \rangle, \qquad f \in B^* \tag{1.6}$$

then $\partial\Psi\ (e) = \partial\Phi\ (e) + f$.

Let us consider the problem of finding a u from E such that $O \in \partial\Phi\ (u) = A\ (u)$ (Problem 1), i.e., the problem of the existence of an L in $\partial\Phi\ (u)$ such that $\langle L,\ h \rangle = 0$, $\forall h \in H$. It has been shown (see [3]), that Problem 1 is equivalent to the problem of the minimum of the functional $\Phi\ (e)$. Namely, the u required is an element from E at which Φ reaches the lower bound on E.

Let us examine a somewhat more general problem (Problem 2). Let C be a multivalued mapping of E in B^*. Find the u from E such that a functional L will be found in $C\ (u)$ for which $\langle L,\ h \rangle = 0$, $\forall\ h \in H$. We represent C in the form $C = A + R$, where $A\ (e) = \partial\Phi\ (e)$ and let $A\ (E) = B^*$. Then, in general a multivalued operator $A^{-1} : B^* \rightarrow E \subset B$ is defined on B^*. We seek the solution of Problem 2 in the form $u = A^{-1}L$. We then obtain the following equation to find L:

$$L + RA^{-1}L = O, \quad O \in B^* \tag{1.7}$$

An extensive literature is devoted to the investigation of the solvability of (1. 7). For instance, if RA^{-1} is a compression operator, then (1. 7) is solvable. We note that the construction of A^{-1} can be considered as a variational problem.

Let us make the form of the functionals $\Phi(e)$ specific. Let ω be a bounded measurable set in R^n and let $\varphi(\mathbf{x}, \mathbf{e})$ be a function in $\omega \times R^m$ that has the properties

$\varphi(\mathbf{x}, \mathbf{e}) > 0$ for $|e| > 0$, $\varphi(\mathbf{x}, \mathbf{e}) = 0$ for $|\mathbf{e}| = 0$ $|\mathbf{e}|$ is the norm in R^m

$\varphi(\mathbf{x}, \mathbf{e},) \leqslant C |e|^p$ for $|\mathbf{e}| \geqslant 1$ $(1 \leqslant p < \infty)$;

for each fixed $\mathbf{e}$ the function $\varphi(\mathbf{x}, \mathbf{e})$ is continuous almost everywhere in $\mathbf{x}$, and for each $\mathbf{x}$ function $\varphi(\mathbf{x}, \mathbf{e})$ is convex in $\mathbf{e}$.

Let $L_p^m(\omega)$ denote a space of measurable vector-functions $\mathbf{e}(\mathbf{x}) = (e_1(\mathbf{x}), \ldots, e_m(\mathbf{x}))$ for which the integral of $|\mathbf{e}(\mathbf{x})|^p$ with respect to ω is finite. It follows [4] from the mentioned properties of the function $\varphi(\mathbf{x}, \mathbf{e})$ and the boundedness of ω that the functional

$$\Phi(\mathbf{e}(\mathbf{x})) = \int_\omega \varphi(\mathbf{x}, \mathbf{e}(\mathbf{x}))\, d\omega \tag{1.8}$$

is defined on $L_p^m(\omega)$.

From the theorem on the general form of a linear continuous functional in $L_p(\omega)$ (see [5]) we have the following representation for $L(e)$ from (1. 1) when $\Phi(e)$ has the form (1. 8):

$$\langle L(e), h\rangle = \int_\omega \sigma(\mathbf{x})\mathbf{h}(\mathbf{x})\, d\omega, \quad \forall \mathbf{h}(\mathbf{x}) \in L_p^m(\omega) \tag{1.9}$$

Condition (1. 1) can be written in the form

$$\int_\omega [\varphi(\mathbf{x}, \mathbf{e}(\mathbf{x}) + \mathbf{h}(\mathbf{x})) - \varphi(\mathbf{x}, \mathbf{e}) - \sigma(\mathbf{x})\mathbf{h}(\mathbf{x})]\, d\omega \geqslant 0, \forall \mathbf{h}(\mathbf{x}) \in L_p^m(\omega) \tag{1.10}$$

L e m m a 1. 3. If $\sigma(\mathbf{x})$ satisfies the condition (1. 10), then for almost all $\mathbf{x}$ from ω following inequality is satisfied

$$\varphi(\mathbf{x}, \mathbf{e}(\mathbf{x}) + \mathbf{h}) - \varphi(\mathbf{x}, \mathbf{e}(\mathbf{x})) \geqslant \sigma(\mathbf{x})\mathbf{h}, \ \forall \mathbf{h} \in R^m \tag{1.11}$$

P r o o f. If the assertion in the lemma is false, then for some $\mathbf{h}$ from R^m there exists a set M of positive measure in ω , on which the opposite inequality to (1. 11) is satisfied. Setting $\mathbf{h}(\mathbf{x})$ equal to $\bar{\mathbf{h}}$ in M and equal to zero outside M , we arrive at a contradiction to (1. 10).

Therefore, it is sufficient to know the expression for the subdifferential (support functions) φ to describe the subdifferential (1. 8). It is convenient to describe the subdifferential φ by using the conjugate (dual to Legendre (Young)) function φ^*

(see [6]),

$$\varphi^*(\mathbf{x}, \sigma) = \sup_{\mathbf{e}\in R^m}(\mathbf{e}\sigma - \varphi(\mathbf{x}, \mathbf{e})), \quad \mathbf{e}\sigma = \sum_i e_i\sigma_i \tag{1.12}$$

The operation of going from φ to φ^* is involutory [6] (for example, if φ, φ^* are continuous everywhere), i.e.,

$$\varphi(\mathbf{x}, \mathbf{e}) = \sup_{\sigma\in R^m}(\mathbf{e}\sigma - \varphi^*(\mathbf{x}, \sigma)) \tag{1.13}$$

Formulas (1.12) and (1.13) define the multivalued mappings $\mathbf{e}(\sigma)$, $\sigma(\mathbf{e})$. Namely $\mathbf{e}(\sigma)$ is the set of all $\mathbf{e}$ from R^m for which the upper bound is reached in (1.12). The $\sigma(\mathbf{e})$ are determined analogously from (1.13). It follows from (1.13) that the support functional to (1.8) allows the representation (1.9), where for almost all $\mathbf{x}$ from ω

$$\sigma(\mathbf{x}) \in \sigma(\mathbf{e}(\mathbf{x})) \tag{1.14}$$

For example, if the function $\varphi(\mathbf{x}, \mathbf{e})$ satisfies the above-mentioned conditions and is smooth for $|\mathbf{e}| > 0$, then (1.14) is equivalent to the relation

$$\sigma(\mathbf{x}) = \nabla_{\mathbf{e}}\varphi(\mathbf{x}, \mathbf{e}(\mathbf{x})), \quad |\mathbf{e}| > 0; \quad \varphi^*(\mathbf{x}, \sigma(\mathbf{x})) = 0, \quad |\mathbf{e}| = 0 \tag{1.15}$$

for almost all $\mathbf{x}$ from ω.

2. On certain stationary problems for viscoplastic media. Let ω be a domain in R^3 filled with a continuous medium. Let us examine the linear manifold U of kinematically admissible velocity fields in ω. The principle of virtual power (see [7] for instance) is the following:

$$\int_\omega \rho\mathbf{h}\frac{d\mathbf{u}}{dt}\, d\omega + \int_\omega \sum_{ij} \sigma_{ij}h_{ij}\, d\omega - F(\mathbf{h}) = 0, \quad \forall\mathbf{h} \tag{2.1}$$

$$h_{ij} = \frac{1}{2}\left(\frac{\partial h_i}{\partial x_j} + \frac{\partial h_j}{\partial x_i}\right)$$

where $F(\mathbf{h})$ is the power of the external forces in the variation $\mathbf{h}$ of the field $\mathbf{u}$ from U, σ_{ij} is the real stress field, and ρ is the density of the medium.

The model of an incompressible viscoplastic medium is defined by the dissipative potential $\varphi(\mathbf{x}, e_{ij})$ (see [8]). Let the function $\varphi(\mathbf{x}, \mathbf{e})$, $\mathbf{e} = (e_{ij})$ possess the properties mentioned in Sect. 1

$$\varphi > c|\mathbf{e}| \left(|\mathbf{e}| = \left(\sum_{ij} e_{ij}^2\right)^{1/2}\right)$$

and let φ be a smooth function for $|\mathbf{e}| > 0$. Then the relationship between σ_{ij} and $\mathbf{u}$ for a viscoplastic medium is defined by the formulas

$$\sigma_{ij} = \frac{\partial\varphi}{\partial e_{ij}}(\mathbf{x}, e_{ij}), \quad |\mathbf{e}| > 0\left(e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) \tag{2.2}$$
$$\varphi^*(\mathbf{x}, \sigma_{ij}) = 0, \quad |\mathbf{e}| = 0$$

Moreover, it is assumed that $\mathbf{u}$ from U satisfies the incompressibility condition $\operatorname{div} \mathbf{u} = 0$.

The flow condition ordinarily (see [9, 10], for example) defines a prism in the space σ_{ij}. The second condition in (2.2), which is also a flow condition, defines a sphere-type domain. Both formulations of the flow condition are evidently equivalent because of the incompressibility condition.

Now, let us examine the principle of virtual power in the following approximation (slow stationary motions):

$$\int_\omega \sum_{ij} \sigma_{ij} h_{ij}\, d\omega = F(\mathbf{h}) \tag{2.3}$$

and we consider the question of the solvability of the problem (2.2), (2.3).

Let us introduce a functional semi-bounded from below in U

$$I(\mathbf{u}) = \int_\omega \varphi(\mathbf{x}, e_{ij}(\mathbf{x}))\, d\omega - F(\mathbf{u}) \tag{2.4}$$

Here $F(\mathbf{u})$ is a linear functional.

Theorem 2.1. If $\mathbf{u}(\mathbf{x})$, $\sigma_{ij}(\mathbf{x})$ are a solution of the problem (2.2), (2.3) then $\mathbf{u}(\mathbf{x})$ minimizes (2.4) in U. If $\mathbf{u}(\mathbf{x})$ minimizes (2.4) in U then there exist $\sigma_{ij}(\mathbf{x})$ such that $\mathbf{u}$, σ_{ij} is the solution of the problem (2.2), (2.3).

The assertion of the theorem results directly from a remark relative to the form of the subdifferential for (1.6), from (1.15), and from the remark in Sect. 1 relative to the solvability of the problem 1.

Thus, the investigation of the correctness of the problem (2.2), (2.3) is reduced to the problem of the minimum of the functional (2.4), which is considered in detail in [8, 11].

Conditions (2.2) show that the problem of steady motions of a viscoplastic medium in a problem in a domain with unknown boundaries. Theorem 2.1 sets up an equivalence between this problem and the problem of the minimum of the functional (2.4), which is generally non-differentiable. Let us note a particular case when the connection between the differential formulation of the problem and the variational principle is especially simple. Let the solution of the problem (2.2), (2.3) be such that $|\mathbf{e}(\mathbf{x})| \geqslant a > 0$ everywhere in ω. In this case, the non-differentiability of $\varphi(\mathbf{x}, e_{ij})$ for $|\mathbf{e}| = 0$ is inessential and the functional (2.4) is differentiable on such an extremal. Under these assumptions, the first condition in (2.2) and (2.3) are the usual Euler equation for the functional (2.4). Namely, under these assumptions the relationship between the equations of motion and the variational principle is set up in [12] for a dissipative potential of the form

$$\varphi(\mathbf{x}, e_{ij}) = \mu I_2^2 / 2 + \tau_0 I_2, \quad I_2 = \Big(\sum_{ij} e_{ij}^2\Big)^{1/2} \tag{2.5}$$

We note that the correctness of a somewhat more general class of problems can be investigated. Namely, let us replace the condition (2.2) by the following relationship:

$$\sigma_{ij} = \frac{\partial \varphi}{\partial e_{ij}} + \rho_{ij}(\mathbf{x}, \mathbf{e}), \quad |\mathbf{e}| > 0 \tag{2.6}$$

$$\varphi^*(\mathbf{x}, \sigma_{ij}) = 0, \quad |\mathbf{e}| = 0$$

The problem (2.3),(2.6) is problem 2 from Sect. 1, where

$$\langle R(\mathbf{u}), \mathbf{h}\rangle = \int_\omega \sum_{ij} \rho_{ij} h_{ij}\, d\omega$$

For example, if $\varphi(\mathbf{x}, \mathbf{e})$ has the form (2.5), the ρ_{ij} satisfy the conditions

$$\rho_{ij}(\mathbf{x}, \mathbf{o}) = 0; \ |\rho_{ij}(\mathbf{x}, \mathbf{e}_1) - \rho_{ij}(\mathbf{x}, \mathbf{e}_2)| \leqslant k_{ij}\, |\mathbf{e}_1 - \mathbf{e}_2|$$

and the k_{ij} are sufficiently small, then the operator RA^{-1} is compressive.

3. **Functions with values in Banach spaces.** We present preparatory material which will be used in Sect. 4 in the investigation of nonstationary problems.

Let B be a separable, reflexive Banach space (see [5]). We consider the function $f(t)$ in the segment $[0, T]$ with values in B; the function $f(t)$ is measurable (see [13], p. 765) if for any ε, $\varepsilon > 0$ there exists a compact K_ε, $K_\varepsilon \subset [0,T]$ of Lebesgue measure less than ε such that the function $f(t)$ is continuous outside K_ε. We shall use the notation $M_B{}^p[0,T]$, $p \geqslant 1$ (see [13]) for the space of measurable functions $f(t)$ for which $\|f\|_B{}^p(t)$ is summable in $[0,T]$. Let $M_B{}^\infty[0, T]$ denote the space of measurable functions for which the quantity $\|f\|_B(t)$ is bounded in $[0,T]$.

We examine the separable Hilbert space H. Let $f(t)$ be a function in $[0, T]$ with values in H. The derivative df/dt is called an element of H such that

$$\lim_{\Delta t\to 0} \| df/dt - (f(t+\Delta t) - f(t))/\Delta t \|_H = 0 \tag{3.1}$$

We will say that the function $f(t)$ has a generalized derivative $f'(t)$ in $[0,T]$ if $f(t)$ is continuous in $[0,T]$, $f'(t)$ is from $M_H{}^1[0, T]$, and

$$\int_0^T (f', \varphi)_H\, dt = -\int_0^T \Big(f, \frac{d\varphi}{dt}\Big)_H dt, \quad \forall \varphi(t) \tag{3.2}$$

where $\varphi(t)$ is from $C_H[0, T]$ and has a piecewise-continuous derivative in $[0,T]$, $\varphi(0) = \varphi(T) = 0$. It can be shown that the generalized derivative agrees with the derivative (3.1) in $[0, T]$, except perhaps in a set of Lebesgue measure zero (see [14]).

An integration by parts formula

$$\int_0^T (f_1', f_2)_H\,dt + \int_0^T (f_1, f_2')_H\,dt = (f_1(T),\ f_2(T))_H - (f_1(0),\ f_2(0))_H$$

holds for functions with values in H.

Let $L_B{}^p[0, T]$ be the factor space of $M_B{}^p[0,\ T]$ in the space of negligible functions (i. e., equal zero almost everywhere) [13]; $L_B{}^p\ [0,\ T]$ is a Banach space with the norm

$$\left\{\int_0^T \|f\|_B^p(t)\,dt\right\}^{1/p}, \quad 1 \leqslant p < \infty; \qquad \inf_M \sup_{t \in [0,\ T]\setminus M} \|f\|_B(t)$$

where M is a set of Lebesgue measure zero in $[0,\ T]$.

L e m m a 3 . 1. Let $\{u_n(t)\}$ be a sequence of piecewise-linear continuous functions, where $\|du_n/dt\|_H \leqslant c$ (the derivative exists everywhere in $[0,\ T]$ except at a finite number of points). Let the function $u_n(t)$ converge uniformly to $u(t)$ in $[0,\ T]$. Then there exists a $u'(t)$ from $M_H{}^\infty[0,\ T]$ and

$$\int_0^T \left(\frac{du_n}{dt},\ v\right)_H dt \to \int_0^T (u',\ v)_H\,dt, \qquad \forall v(t) \in M_H^1[0,\ T]$$

The assertion in the lemma follows from the fact that $L_{B^*}^\infty[0,\ T]$ is conjugate to $L_B{}^1\ [0,\ T]$ [13] and the properties of weak compactness of the space conjugate to the separable space [5].

Let $R(t,\ e)$ be a number function defined in $[0,\ T] \times B$, $R(t,\ 0) = 0$, and $R(t,\ e(t))$ is from $M^\infty[0,\ T]$ for $e(t)$ from $M_B{}^\infty[0,\ T]$.

L e m m a 3.2. If

$$\int_0^T R(t, e(t))\,dt = 0 \quad \left(\int_0^T R(t,\ e(t))\,dt \geqslant 0\right), \quad \forall e(t) \in M_B{}^\infty[0,\ T]$$

then $R(t,\ e) = 0\ (R(t,\ e) \geqslant 0)$ for almost all t from $[0,\ T]$ and all e from B.

The proof of the lemma is analogous to the proof of Lemma 1.3.

L e m m a 3.3. If $e(t)$ from $M_B{}^\infty[0,\ T]$, then a section from $M_{B^*}^\infty[0,\ T]$ is contained in $A(e(t)) = \partial\Phi(e(t))$, i.e., there exists a function $\omega(t)$ from $M_{B^*}^\infty\ [0,\ T]$ such that $\omega(t) \in A(e(t))$.

For the proof, we show that a measurable section is contained in $A(e(t))$. We approximate $e(t)$ by step functions $e_n(t)$ in the norm $L_B{}^p[0,\ T]$ [13]. There exists a step function $\omega_n(t)$ from $M_{B^*}^\infty[0,\ T]$ such that, firstly $\omega_n(t) \in A(e_n(t))$ and secondly

$$\int_0^T \langle \omega_n(t), v(t)\rangle\, dt \leqslant \int_0^T [\Phi(e_n(t) + v(t)) - \Phi(e_n(t))]\, dt$$

The sequence $\{\omega_n(t)\}$ can be considered weakly convergent to $\omega(t)$ from $M^{\infty}_{B^*}[0,T]$ and

$$0 \leqslant \int_0^T [\langle \omega(t), v(t)\rangle - (\Phi(e(t) + v(t)) - \Phi(e(t)))]\, dt, \tag{3.3}$$

$$\forall v(t) \in M^{\infty}_{B}[0, T]$$

The assertion in Lemma 3.3 follows from (3.3) and Lemma 3.2.

4. On certain nonstationary formulations of problems in the theory of viscoplastic media. Let us first consider the problem concerning the solvability of an abstract parabolic equation with a multi-valued stationary operator, and then let us apply the results obtained to the investigation of dynamic problems for viscoplastic media.

Let $u(t)$ be from $C_H[0, T] \cap M_B{}^{\infty}[0, T]$ and $u'(t)$ from $M_H{}^{\infty}[0, T]$. Furthermore, let $\omega(t)$ from $M^{\infty}_{B^*}[0, T]$ be the section $A(u(t))$ (the operator $A = \partial\Phi$ is introduced in Sect. 1, and for simplicity is assumed independent of t). Assume that $f(t)$ is from $C_H[0, T]$. The function $u(t)$ is called the generalized solution of the nonstationary problem

$$A(u(t)) + \partial u / \partial t = f(t), \qquad u(0) = u_0$$

if for almost all t from $[0, T]$ the equality

$$(u', v)_H + \langle \omega(t), v\rangle = (f, v)_H, \quad u(0) = u_0, \quad \forall v \in B \cap H \tag{4.1}$$

is satisfied.

The uniqueness theorem for the generalized solution is proved by the same scheme as for the single-valued monotonic operators (see [15], p. 173). The existence theorem for the generalized solution can be obtained on the basis of [16].

Namely, let $\{\Delta t_i^n\}$, $i = 1, \ldots, n$; $\Sigma_i \Delta t_i^n = T$ be the partition of $[0, T]$. We examine the chain of elements u_i^n from $B \cap H$ such that

$$\inf_u \left\{\frac{1}{2\Delta t_i^n} \| u - u_{i-1}^n \|_H{}^2 + \Phi(u) - (f(t_i^n), u)_H\right\} = \tag{4.2}$$

$$\frac{1}{2\Delta t_i^n} \| u_i^n - u_{i-1}^n \|_H^2 + \Phi(u_i^n) - (f(t_i^n), u_i^n)_H$$

There follows from the Morrow—Rockafellar theorem (see Sect.1) and (4.2) that there exist $\omega(u_i^n)$ from $\partial\Phi(u_i^n)$ for which

$$\left(\frac{u_i^n - u_{i-1}^n}{\Delta t_i^n}, v\right)_H + \langle\omega(u_i^n), v\rangle = (f(t_i^n), v)_H, \quad \forall v \in B \cap H \tag{4.3}$$

It is proved in [16] that

$$\|u_i^n\|_B + \|u_i^n\|_H \leqslant c_1 \tag{4.4}$$

Let $f(t)$ and the initial element u_0 satisfy the conditions

$$\|f(t_1) - f(t_2)\|_H \leqslant c_2 |t_1 - t_2| \max_{i=1,2} \|f(t_i)\|_H \tag{4.5}$$

$$\|(u_1^n - u_0)/\Delta t_1^n\|_H \leqslant c_3, \quad \forall \Delta t_1^n \tag{4.6}$$

Under these assumptions, it is proved in [16] that

$$\|(u_i^n - u_{i-1}^n)/\Delta t_i^n\|_H \leqslant c_4, \quad i = 2, 3, \ldots, n \ (\forall n) \tag{4.7}$$

Let us introduce the functions $u^n(t)$, $\bar{u}^n(t)$, where $u^n(t)$ is a step function equal to u_{i-1}^n for $t_{i-1}^n \leqslant t < t_i^n$, and $\bar{u}^n(t)$ is a continuous piece-wise linear function equal to u_i^n for $t = t_i^n$. We find from (4.3) that

$$\int_0^T \left(\frac{d\bar{u}^n}{dt}, v(t)\right)_H dt + \int_0^T \langle\omega(u^n(t)), v(t)\rangle\, dt = \tag{4.8}$$

$$\int_0^T (f^n(t), v(t))_H\, dt, \quad \forall v(t) \in M^1_{B\cap H}[0, T]$$

$$\omega(u^n(t)) = \omega(u_{i-1}^n), \quad f^n(t) = f(t_{i-1}^n) \quad (t_{i-1}^n \leqslant t < t_i^n)$$

It is proved in [16] that upon compliance with conditions (4.5) and (4.6), the functions $\bar{u}^n(t)$ converge in the norm $C_H[0, T]$ to $u(t)$. It follows from Lemma 3.1 and (4.7) that there exists a $u'(t)$. Because of (4.4) it can be considered that the functions $\omega(u^n(t))$ converge weakly to $\chi(t)$ in $M^\infty_{B^*}[0, T]$.

We show that $\chi(t)$ is from $A(u(t))$. The equality

$$\int_0^T [(u', v)_H + \langle\chi, v\rangle - (f, v)_H]\, dt = 0, \quad \forall v \in M'_{B\cap H}[0, T] \tag{4.9}$$

follows from (4.8).

Let $v(t)$ be from $M^\infty_B[0, T]$, $\omega(v(t))$ from $M^\infty_{B^*}[0, T]$ and $\omega(v(t))$ from $A(v(t)) = \partial\Phi(v(t))$ (see Lemma 3.3). Furthermore (analogously to [15]:

$$\int_0^T \langle \omega(u^n(t) - \omega(v(t)), \quad u^n(t) - v(t)\rangle\, dt \geqslant 0, \quad \forall v(t) \in M_B{}^\infty[0, T] \tag{4.10}$$

From (4.8) we find for $v(t) = u^n(t)$

$$\int_0^T \langle \omega(u^n(t), u^n(t)\rangle\, dt = \int_0^T \left[(f^n(t), u^n(t))_H - \left(\frac{d\bar{u}^n}{dt}, u^n\right)_H\right] dt = \tag{4.11}$$

$$\frac{1}{2}\|u_0\|_H{}^2 - \frac{1}{2}\|u_n{}^n\|_H{}^2 + \frac{1}{2}\sum_{i=1}^{n}\|u_i{}^n - u_{i-1}^n\|_H{}^2 + \int_0^T (f^n(t), u^n(t)_H dt \to$$

$$\frac{1}{2}\|u_0\|_H{}^2 - \frac{1}{2}\|u(T)\|_H{}^2 + \int_0^T (f(t), u(t))_H\, dt$$

From (4.9) –(4.11) and the integration by parts formulas in Sect. 3, we have

$$0 \leqslant \int_0^T \langle \chi - \omega(v(t)), \quad u - v\rangle\, dt, \quad \forall v(t) \in M_B{}^\infty[0, T]$$

It follows from the last inequality, Lemmas 3.2 and 1.2 that χ is from $A(u(t))$, i.e., the existence of the generalized solution (4.1) is proved.

Parabolic equations with multivalued stationary operators A have been examined in [17, 18]. The concept of the generating operator of a semigroup and the concept of the generalized solution itself, which is weaker compared to (4.1), were used to construct the generalized solution in [17]. The existence of an ordinary generalized solution was proved successfully above because of the introduction of condition (4.6) on the initial element u_0. (Such a condition on the initial element was not considered in [17, 18]). It is shown in [16] that compliance with condition (4.6) is associated with the requirement of definite smoothness of the initial conditions, in particular, it is always satisfied if the initial element u_0 minimizes $\Phi(u)$, for example.

We note that the variational scheme elucidated for the construction of approximate solutions of parabolic equations (agreeing substantially with the Rote scheme) can be of interest from the computational viewpoint even in the case of linear equations. This is related to the fact that by using dual functionals the upper and lower estimates of the minimal values of the functionals (4.2) can effectively be obtained, which affords additional information about the accuracy of the approximate solution as compared to the method of grids, for instance.

Now, we turn to the nonstationary motions of a viscoplastic medium. Let us consider the principle of virtual powers (2.1) in the following approximation (slow nonstationary motion):

$$\int_\omega \left(\rho\frac{\partial \mathbf{u}}{\partial t}\mathbf{h} + \sum_{ij}\sigma_{ij}h_{ij}\right) d\omega = F(\mathbf{h}), \quad \operatorname{div}\mathbf{u} = 0, \ \mathbf{u}\big|_{t=0} = \mathbf{u}_0(x) \tag{4.12}$$

where the σ_{ij} are determined by (2.2). Under these conditions, the problem (4.12) is a particular case (4.1). Namely

$$\langle L(\mathbf{u}), \mathbf{h}\rangle = \int_\omega \sum_{ij} \sigma_{ij} h_{ij}\, d\omega; \quad (\mathbf{u}, \mathbf{h})_H = \int_\omega \rho \sum_i u_i h_i\, d\omega$$

Let us note that sufficiently many specific mechanical problems concerning the nonstationary motions of a viscoplastic medium [19] has been considered. However, the correct mathematical formulation, taking account of interaction between rigid zones and flow domains, has not always been given.

5. On the limit load factor for a rigidly plastic medium. We consider the application of Theorem 2.1 to the problem of bilateral estimates of the limit load factor.

In the case of a rigidly plastic medium the dissipative potential $\varphi(\mathbf{x}, e_{ij})$ is such that $\varphi(\mathbf{x}, \lambda e_{ij}) = \lambda\, \varphi(\mathbf{x}, e_{ij})$, $\forall\lambda \geqslant 0$ [8].

Let the kinematically allowable velocity fields form a linear space. The limit load factor c^* for the external forces with volume density $\mathbf{f}$ and surface density $\mathbf{t}$ is determined by the formula [8]

$$(c^*)^{-1} = \sup_{\mathbf{u}(\mathbf{x})\in U} F(\mathbf{u}(\mathbf{x})) \left[\int_\omega \varphi(\mathbf{x}, e_{ij}(\mathbf{x}))\, d\omega\right]^{-1}$$

$$F(\mathbf{u}) = \int_\omega \mathbf{f}\mathbf{u}\, d\omega + \int_{\partial\omega} \mathbf{t}\mathbf{u}\, dS$$

If $F(\mathbf{u}) > 0$ and for a certain $\mathbf{u}$ from U, then we obtain the upper bound for c^*.

$$c^* \leqslant c_u = \left(\int_\omega \varphi(\mathbf{x}, e_{ij}(\mathbf{x}))\, d\omega\right) / F(\mathbf{u})$$

Let c_σ be a nonnegative number such that functions σ_{ij} from $M^\infty(\omega)$, exist for which the following equalities are satisfied

$$\int_\omega \sum_{ij} \sigma_{ij} e_{ij}\, d\omega = c_\sigma F(\mathbf{u}),\ \forall \mathbf{u}(\mathbf{x}) \in U,\ \varphi^*(\mathbf{x}, \sigma_{ij}(\mathbf{x})) = 0 \tag{5.1}$$

Since $\varphi^*(\mathbf{x}, \sigma_{ij}) = 0$ then

$$\sum_{ij} \sigma_{ij} e_{ij} \leqslant \varphi(\mathbf{x}, e_{ij})$$

and we obtain from (5.1) that

$$\frac{1}{c_\sigma} \geqslant F(\mathbf{u}) \Big/ \int_\omega \varphi\, d\omega, \quad \forall \mathbf{u} \in U$$

Therefore, if $F(\mathbf{u}) > 0$ for at least one $\mathbf{u}$ from U, then $c^* \geqslant c_\sigma$
Let $c_* = \sup c_\sigma$. It is evident that

$$c^* \geqslant c_* \qquad (5.2)$$

Theorem 5.1. $c^* = c_*$

Proof. It can be considered that $c^* > 0$ since if $c^* = 0$, then we can take $\sigma_{ij} = 0$ for $c_\sigma = 0$. Let us take the positive number c, $0 < c < c^*$ and let us consider the functional

$$J_c(\mathbf{u}) = \int_\omega \varphi(\mathbf{x}, e_{ij}(\mathbf{x}))\, d\omega - cF(\mathbf{u})$$

Evidently $J_c(\mathbf{u}) \geqslant 0$ for all $\mathbf{u}$ from U, i.e., $\mathbf{u} = \mathbf{0}$ is a vector field minimizing $J_c(\mathbf{u})$. But then according to Theorem 2.1 there exist σ_{ij} such that (5.1) is satisfied for $c_\sigma = c$.

The theorem is proved. Only the inequality (5.2) was known earlier in the general case.

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